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Passive locomotion via normal-mode coupling in a submerged spring–mass system

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The oscillations of a class of submerged mass–spring systems are examined. An inviscid fluid model is employed to show that the hydrodynamic effects couple the normal modes of these systems. This coupling of normal modes can excite the displacement mode – yielding passive locomotion of the system – even when starting with zero displacement velocity. This is in contrast with the fact that under similar initial conditions but without the hydrodynamic coupling, such systems cannot achieve a net displacement. These ideas are illustrated via two examples of a two-mass and a three-mass system.

Key words: low-dimensional models, nonlinear dynamical systems

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1. Introduction

This paper is concerned with the dynamics of a submerged one-dimensional harmonic chain consisting of interconnected springs and masses (see figure 1). We show that the surrounding fluid couples the normal modes of the submerged chain. We also show that starting with zero displacement velocity and under no applied forces, the submerged chain can undergo passive locomotion by a proper initial excitation of its oscillation modes.

Our primary motivation for studying this problem stems from our interest in understanding the swimming of aquatic animals, particularly whether the passive coupling between the animals' elastic properties and the surrounding flow plays a role in enhancing their locomotion. Indeed, fish do not swim steadily, and many fish use a 'burst-and-coast' behaviour of alternating accelerations (burst) with passive glides (coast). The burst-and-coast behaviour has been shown to minimize energy in Videler & Weihs (1982), and experimental evidence exists that the elastic properties of fish are tuned to hydrodynamic forces to help them maximize their passive locomotion (see Beal *et al.* 2006). In this work, we propose the submerged mass–spring systems as useful prototypes – and not necessarily accurate models of biological swimmers – that allow one to study the passive coupling between the system's elasticity and the surrounding fluid subject to initial conditions only (similar to the coast phase in fish swimming) and to explore the effect of this coupling on the otherwise known dynamics of the linear harmonic chains.

We consider n identical circular cylinders of equal radius a such that the centres of every two adjacent cylinders are connected by a linear spring of constant stiffness

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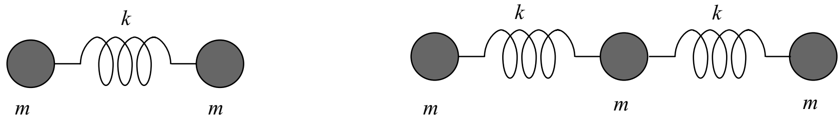

(a) Two-mass system (b) Three-mass system

FIGURE 1. Identical bodies connected via linear springs and constrained to move along the line connecting their centres. Each mass–spring system is submerged in an infinite volume of incompressible inviscid fluid. The hydrodynamic effects couple the normal modes of oscillations of these systems but in such a manner that starting with zero displacement velocity, the system in (a) cannot undergo a net displacement while that in (b) can.

k and undeformed length l (see figure 1). The cylinders are constrained to move along the line joining their centres. The mass–spring system is submerged in an infinite volume of inviscid, incompressible fluid, and the cylinders are considered to be neutrally buoyant. Further, the system is placed initially in potential flow (no vorticity), and the flow is assumed to remain irrotational for all time, such that the effect of the fluid is accounted for using the added-mass effect. By way of background, note that in any system of bodies submerged in potential flow, as the bodies accelerate, they also accelerate the surrounding fluid, creating additional forces on the surfaces in contact with the fluid. These forces are accounted for by 'adding mass' to the submerged bodies as reviewed comprehensively in Brennen (1982). Also by way of background, note that the inviscid-fluid assumption is widely employed in modelling swimming at large Reynolds numbers. Kanso *et al.* (2005) showed that a deformable body can swim in an inviscid fluid even in the absence of vorticity (within the potential flow model) because of the added-mass effect only. Kanso (2009) proved that this model for swimming in potential flow is related to Lighthill's reactive force theory (Lighthill 1975) for swimming of large fish, where Lighthill has argued that for this class of aquatic animals, the fluid added-mass effect is significant. These works, including Lighthill's theory, are concerned with active swimming because of prescribed deformations, while the focus of the current paper is on the passive dynamics and locomotion of the submerged spring–mass systems. We show that the hydrodynamic coupling introduces nonlinear terms in the equations governing the motion of these submerged systems that lead to non-trivial coupling of their normal modes. The resulting problem is reminiscent of the Fermi–Pasta–Ulam (FPU) problem that played a historic role in the discovery of nonlinear resonance and subsequently has had an enormous impact on the field of nonlinear dynamics (see Ford 1992 for a pedagogical discussion of the FPU problem). The difference between the submerged system studied here and the FPU system is that the nonlinear coupling in the latter is introduced via weakly nonlinear springs, while in the submerged system, the nonlinear coupling is due to the presence of the fluid and, unlike in the FPU problem, can excite the displacement mode even when starting with zero displacement velocity.

For concreteness, let the coordinates of the centres of the cylinders, measured from the undeformed configuration, be labelled as $x_1, x_2, \ldots, x_n$ and introduce the n-dimensional position vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, where $()^T$ denotes the transpose. The Lagrangian governing the one-dimensional motion of the submerged mass–spring system is given by

$$L = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{M}\dot{\boldsymbol{x}} - \frac{1}{2}\boldsymbol{x}^T \boldsymbol{K}\boldsymbol{x}, \tag{1.1}$$

where $\boldsymbol{K}$ is the $n \times n$ stiffness matrix whose entries depend on the spring constant k and $\boldsymbol{M}$ is an $n \times n$ symmetric matrix denoting the mass plus the added mass of the submerged bodies. In the potential flow regime, the effect of the fluid is completely accounted for by using the added-mass effect, the values of which depend not only on the shape of the submerged bodies, here considered to be circular cylinders, but also on their position relative to each other, which can be parameterized by the $n-1$ variables $(x_{i+1}-x_i)$, $i=1,\ldots,n$ (for more details, see, e.g., Kanso *et al.* 2005; Nair & Kanso 2007).

In the absence of applied forces, this class of problems is energy preserving; the total energy $E=(1/2)\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} + (1/2)\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x}$ is conserved. It also admits a momentum integral of motion; namely the momentum of the solid–fluid system associated with a rigid or net displacement of the system is conserved (as verified in §§ 2 and 3 via examples). This momentum conservation reflects the fact that the system is invariant to rigid transformations along the line joining their centres. This invariance is a consequence of the mass matrix and the fact that the spring forces is a function of relative positions only.

Without the hydrodynamic coupling effects, that is to say in the case in which the mass matrix is diagonal with equal and constant entries, these systems are linear, and their dynamics is completely understood in terms of their normal modes and associated frequencies (i.e. eigenvectors and eigenvalues of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}$ for $\boldsymbol{M}$ diagonal with equal entries). Typically, the smallest eigenvalue is zero, and its associated normal mode corresponds to a net displacement of the whole system. For the submerged systems considered here, the dynamics is not linear – indeed, while the springs couple the dynamics of the masses linearly, the presence of the fluid couples their dynamics nonlinearly. Our approach to investigating the effect of the hydrodynamic coupling on the response of these systems is to project their dynamics on to the normal modes obtained when there is no hydrodynamic coupling. In § 2, we use the example of a two-mass chain to show that although the hydrodynamic coupling alone cannot cause a net displacement of the system when starting with zero displacement velocity, it does act as a periodic forcing of the oscillation mode, leading to a dynamic response of the submerged system similar to that observed in parametrically excited systems. It is worth noting that the interactions of two bodies in potential flow was also considered in different contexts in the recent works of Burton, Gratus & Tucker (2004), Wang (2004) and Crowdy, Surana & Yick (2007). In § 3, we show that for the three-mass chain, the coupling of the normal modes through the hydrodynamic effects can excite the displacement mode even when starting with zero displacement velocity, allowing the system to undergo passive locomotion.

2. Submerged two-mass system

Consider the example shown in figure 1(a) of two submerged masses connected via a linear spring. The mass matrix $\boldsymbol{M}=\boldsymbol{M}_{cyl} + \boldsymbol{M}_{added}$ is given as the sum of a diagonal matrix $\boldsymbol{M}_{cyl}$ representing the actual mass of the cylinders and an added-mass matrix $\boldsymbol{M}_{added}$ that accounts for the forces required for the cylinders to accelerate the surrounding fluid,

$$\boldsymbol{M}_{cyl} = \begin{pmatrix} m_{cyl} & 0 \\ 0 & m_{cyl} \end{pmatrix}, \qquad \boldsymbol{M}_{added} = \begin{pmatrix} m_{1\,added} & m_{12} \\ m_{12} & m_{2\,added} \end{pmatrix}. \tag{2.1}$$

The mass of the cylinders is given by $m_{cyl}=\rho_{cyl}\pi a^2$, where the density of the cylinders ρ_{cyl} is equal to the density of the fluid ρ for neutrally buoyant bodies. The three entries $m_{1\,added}$, $m_{2\,added}$ and m_{12} of the added-mass matrix depend nonlinearly on the relative distance $x_2 - x_1$ between the two masses. By symmetry, one has $m_{1\,added} = m_{2\,added} = m_{added}$. Let $m = m_{cyl} + m_{added}$, and write the total mass and stiffness matrices for the submerged system as follows:

$$\boldsymbol{M} = \boldsymbol{M}_{cyl} + \boldsymbol{M}_{added} = \begin{pmatrix} m & m_{12} \\ m_{12} & m \end{pmatrix}, \qquad \boldsymbol{K} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}. \tag{2.2}$$

Without the effect of the fluid (when $\boldsymbol{M}_{added}$ is set to zero), the mass–spring system admits an analytic solution in terms of the normal modes $y_1 = (x_1 + x_2)/\sqrt{2}$ and $y_2 = (x_2 - x_1)/\sqrt{2}$ and their associated frequencies $\omega_1 = 0$ and $\omega_2 = \sqrt{2k/m_{cyl}}$, with y_1 measuring the net displacement of the system and y_2 measuring the length of the deformed spring. Namely one has

$$\left.\begin{aligned} y_1(t)|_{decoupled} &= \dot{y}_1(0)t + y_1(0), \\ y_2(t)|_{decoupled} &= y_2(0)\cos(\omega_2 t) + \frac{\dot{y}_2(0)}{\omega_2}\sin(\omega_2 t), \end{aligned}\right\} \tag{2.3}$$

where $y_{1,2}(0)$, $\dot{y}_{1,2}(0)$ are the initial conditions. Clearly, without the fluid, the two modes y_1 and y_2 are completely decoupled as seen in (2.3). In particular, starting with zero net velocity $\dot{y}_1(0) = 0$, the system cannot undergo a net displacement. This is also true for the submerged two-mass system. To verify this claim, note first that the added masses depend on $x_2 - x_1$ only and not on $x_1 + x_2$. Hence, when rewritten in terms of the normal mode variables y_1 and y_2 and their time derivatives, the Lagrangian function in (1.1) takes the form

$$L(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}(m + m_{12})\dot{y}_1^2 + \frac{1}{2}(m - m_{12})\dot{y}_2^2 - \frac{1}{2}(2k)y_2^2, \tag{2.4}$$

which is independent of y_1. The variable y_1 is referred to as an ignorable coordinate, and the associated momentum $p_1 = \partial L/\partial \dot{y}_1 = (m + m_{12})\dot{y}_1$ is an integral of motion; that is to say it is conserved. In particular, starting with $\dot{y}_1(0) = 0$, the value of $(m + m_{12})\dot{y}_1$ remains identically zero for all time. Now, although m_{12} could be negative, its absolute value for two identical bodies is typically smaller than m, and hence the term $(m + m_{12})$ is strictly positive. Therefore, $\dot{y}_1$ must be zero for all time – which confirms the claim that starting from rest, the hydrodynamic effects within the potential flow model are not sufficient to excite the displacement mode of the two-mass system. The presence of the fluid does however couple the displacement and oscillation modes y_1 and y_2 nonlinearly. To illustrate this point, the equations governing the motion of the submerged two-mass system are now derived using approximate expressions for the added masses. In general, the added masses are given as boundary integrals of the fluid velocity potentials (over the boundaries of the submerged bodies) that are difficult to evaluate explicitly for complex geometries such as the multipe-connected boundaries considered here. (See, for example, Lamb 1932, Brennen 1982 and Nair & Kanso 2007 for explicit expressions of the added masses as boundary integrals of the fluid velocity potentials, and see Crowdy *et al.* 2007 for details on how to compute the fluid velocity potentials for doubly connected fluid domains.) Approximate expressions for the added masses for two circular cylinders have been obtained in the work of Nair & Kanso (2007) based on the assumption that the distance between the two cylinders is large compared with their radius. Following a similar approach and retaining only terms that are up to order $1/(l + x_2 - x_1)^2$, one gets

$$m_{added} = \rho\pi a^2, \qquad m_{12} = -\rho\pi \frac{2a^4}{(l + x_2 - x_1)^2}, \tag{2.5}$$

where the value of m_{added} is equal to the value of the added mass for one circular cylinder submerged in an infinite fluid domain (see e.g. Lamb 1932), and the hydrodynamic coupling between the two cylinders is manifested only through the term m_{12}. These expressions are valid only when the ratio of the cylinders' radius to their separating distance is small $a/(l + x_2 - x_1) \ll 1$ and, in particular, when $a/l \ll 1$. We use l to non-dimensionlize length and the condition $a/l \ll 1$ to introduce the length scaling parameter ϵ such that

$$\epsilon\tilde{a} = \frac{a}{l}, \qquad \epsilon\tilde{x}_{1,2} = \frac{x_{1,2}}{l}, \qquad \epsilon\tilde{y}_{1,2} = \frac{y_{1,2}}{l}, \tag{2.6}$$

where the tilde $\tilde{()}$ denotes the dimensionless counterpart of (). Substitute (2.5) into (2.2) and then divide by $\rho\pi a^2$ to express the entries of the mass matrix $\boldsymbol{M}$ in dimensionless form, which leads, upon employing the scaling law in (2.6), to

$$\tilde{m} = \frac{m}{\rho\pi a^2} = 2, \qquad \epsilon^2\tilde{m}_{12} = \frac{m_{12}}{\rho\pi a^2} = \frac{-2\epsilon^2\tilde{a}^2}{(1 + \epsilon(\tilde{x}_2 - \tilde{x}_1))^2} = \frac{-2\epsilon^2\tilde{a}^2}{(1 + \epsilon\tilde{y}_2\sqrt{2})^2}. \tag{2.7}$$

Finally, introduce a spring stiffness $\tilde{k}$ such that $\tilde{k} = k/\rho\pi a^2$. To this end, the Lagrangian in (2.4) could be rewritten in terms of the properly scaled, dimensionless parameters and normal modes,

$$L(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}(m + \epsilon^2 m_{12})\dot{y}_1^2 + \frac{1}{2}(m - \epsilon^2 m_{12})\dot{y}_2^2 - \frac{1}{2}(2k)y_2^2, \tag{2.8}$$

where we have dropped the tilde notation, considering all variables, except time, are now non-dimensional. Lagrange's equations of motion for the submerged two-mass system take the form

$$(m + \epsilon^2 m_{12})\ddot{y}_1 + \epsilon^2 \frac{\partial m_{12}}{\partial y_2}(\dot{y}_1\dot{y}_2) = 0, \tag{2.9a}$$

$$(m - \epsilon^2 m_{12})\ddot{y}_2 + 2ky_2 - \epsilon^2 \frac{\partial m_{12}}{\partial y_2}\left(\frac{\dot{y}_1^2 + \dot{y}_2^2}{2}\right) = 0. \tag{2.9b}$$

The fact that $(m + \epsilon^2 m_{12})\dot{y}_1$ is an integral of motion for the submerged system can be verified by integrating (2.9*a*) directly. This conserved quantity together with the term $(m + \epsilon^2 m_{12})$ being strictly positive confirms that the displacement mode cannot be excited when starting with $\dot{y}_1(0) = 0$; that is to say $\dot{y}_1$ remains zero for all time. The presence of the fluid does however affect the dynamics of the two-mass system. When $\epsilon = 0$, the submerged system in (2.9) is said to be hydrodynamically decoupled, and its dynamic behaviour is identical to the system in space whose solution is given in (2.3) but with oscillation frequency $\omega_2 = \sqrt{2k/m}$ instead of $\omega_2 = \sqrt{2k/m_{cyl}}$. In other words, the oscillation frequency $\omega_2 = \sqrt{2k/m}$ corresponds to the case in which there is no hydrodynamic coupling between the two cylinders (cylinders only coupled mechanically through the linear spring k), and each cylinder is treated hydrodynamically as if it were alone in the fluid domain. The hydrodynamic coupling between the two masses affects the amplitude and frequency of the oscillations $y_2(t)$ even when $\dot{y}_1 = 0$ for all time, as seen from (2.9*b*). Figure 2 shows the difference between the response $y_2(t)$ of the coupled system in (2.9) and that of the hydrodynamically

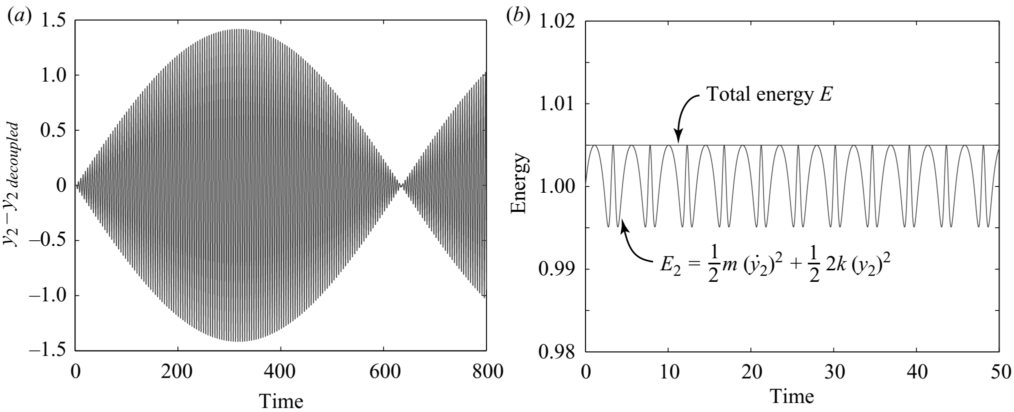

FIGURE 2. Submerged two-mass system. (a) Response of the system in (2.9) relative to the solution given in (2.3) with $\omega_2 = \sqrt{2k/m}$ for the hydrodynamically decoupled system. The initial conditions are chosen to be $y_1 = y_2 = 0$, $\dot{y}_1 = 0$, $\dot{y}_2 = 1$. The parameter values are set to $\epsilon = 0.1$, $k = 2$, $2a^2 = 1$. The integration time is 800. The displacement variable $y_1(t)$ (not shown) is identically zero for the entire duration of the numerical integration. The presence of the fluid causes the amplitude of the oscillations y_2 to vary in time relative to the decoupled system in a way similar to the behaviour of parametrically excited systems. (b) Total energy E of the system in (2.9) is conserved (as expected), while the energy-like quantity $E_2 = (1/2)m\dot{y}_2^2 + (1/2)(2k)y_2^2$ of the oscillation mode varies periodically. The energy because of the hydrodynamic coupling, $E_{coupling} = -(1/2)\epsilon^2 m_{12}\dot{y}_2^2$ (not shown), also varies periodically in time, which means that the hydrodynamic coupling acts as a periodic forcing to the system's parameters; hence the response seen to the left.

decoupled system when the displacement mode y_1 is not excited, $y_1(0) = \dot{y}_1(0) = 0$. As expected, the displacement mode remains identically zero for all time. However, the hydrodynamic coupling causes the amplitude of the oscillations $y_2(t)$ to vary in time relative to those of the hydrodynamically decoupled system in a way similar to the behaviour of parametrically excited systems (see for example Nayfeh 1973). This interesting behaviour can be explained by examining the energy of the system. Note that in general, the total energy E of the two-mass system can be written as

$$E = \underbrace{\frac{1}{2}m\dot{y}_1^2}_{E_1} + \underbrace{\frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}(2k)y_2^2}_{E_2} + \underbrace{\frac{1}{2}\epsilon^2 m_{12}\left(\dot{y}_1^2 - \dot{y}_2^2\right)}_{E_{coupling}}, \tag{2.10}$$

where E_1 can be thought of as the energy associated with the displacement mode, E_2 as that associated with the oscillation mode and $E_{coupling}$ as the energy because of the hydrodynamic coupling between the two cylinders. The total energy E is conserved as argued in § 1 and shown in figure 2(b). However, the energy $E_2 = (1/2)m\dot{y}_2^2 + (1/2)(2k)y_2^2$ associated with the oscillation mode varies periodically. (Note that E_1 is identically zero, since the displacement mode is not excited.) The energy $E_{coupling} = (-1/2)\epsilon^2 m_{12}\dot{y}_2^2$ because of the hydrodynamic coupling also varies periodically in time, which means that the hydrodynamic coupling acts as a periodic forcing to the system's parameters; hence the interesting response seen in figure 2(a). When $\dot{y}_1 \neq 0$, the hydrodynamic effects couple the two normal modes nonlinearly in a way that could be advantageous to the displacement mode as shown in figure 3. The energy exchange between the displacement and the oscillations modes is shown in figure 4.

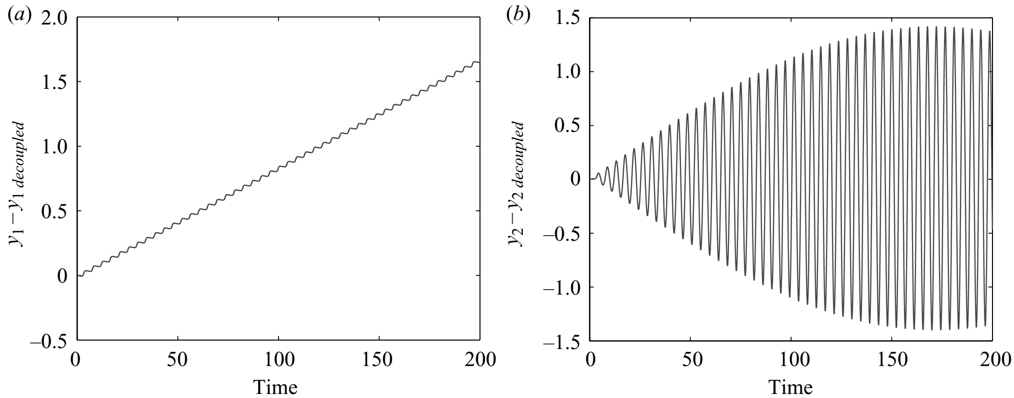

FIGURE 3. Submerged two-mass system: response of the system in (2.9) relative to the solution given in (2.3) with $\omega_2 = \sqrt{2k/m}$ for the hydrodynamically decoupled system. The initial conditions are chosen to be $y_1 = y_2 = 0$, $\dot{y}_1 = \dot{y}_2 = 1$. The parameter values are set to $\epsilon = 0.1$, $k = 2$, $2a^2 = 1$. The integration time is 200.

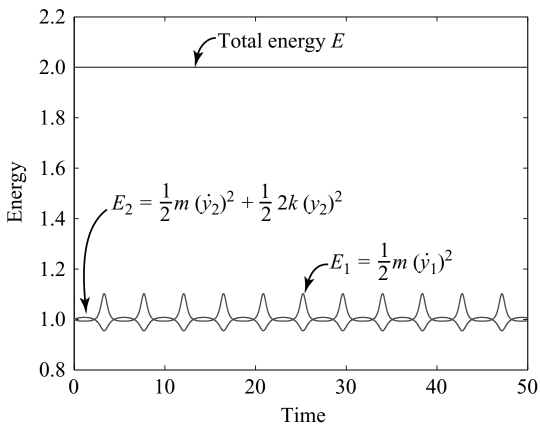

FIGURE 4. Submerged two-mass system: energy plot for the same parameter values as those shown in figure 3. The energies $E_1 = (1/2)m\dot{y}_1^2$ and $E_2 = (1/2)m\dot{y}_2^2 + (1/2)2ky_2^2$ associated with the displacement and oscillation modes, respectively, vary periodically in time such that energy is transferred between the two modes. There is also transfer of energy among E_1, E_2 and the energy because of the hydrodynamic coupling $E_{coupling} = (1/2)\epsilon^2 m_{12}(\dot{y}_1^2 - \dot{y}_2^2)$ (not shown) which varies periodically in time as well, such that the total energy E is conserved.

3. Submerged three-mass system

We now consider the example shown in figure 1(*b*) of three submerged cylinders connected via linear springs. The mass matrix $\boldsymbol{M}$ can be written as the sum of a 3×3 diagonal matrix $\boldsymbol{M}_{cyl}$ with equal entries m_{cyl} plus a 3×3 symmetric added-mass matrix $\boldsymbol{M}_{added}$. In general, the entries of the added-mass matrix $\boldsymbol{M}_{added}$ are nonlinear functions of both $x_2 - x_1$ and $x_3 - x_2$, and its diagonal entries are not necessarily equal. The assumption that the distances between the cylinders are large compared with their radii allows one to argue that the diagonal entries of the added-mass matrix are all equal to $\rho\pi a^2$ and that its off-diagonal terms are of order ϵ^2. Following a scaling argument similar to the one presented in § 2, the mass and stiffness matrices can be written as

$$\boldsymbol{M} = \boldsymbol{M}_{cyl} + \boldsymbol{M}_{added} = \begin{pmatrix} m & \epsilon^2 m_{12} & \epsilon^2 m_{13} \\ \epsilon^2 m_{12} & m & \epsilon^2 m_{23} \\ \epsilon^2 m_{13} & \epsilon^2 m_{23} & m \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}. \tag{3.1}$$

When $\epsilon = 0$, the system is hydrodynamically decoupled, and its dynamic response can be expressed analytically in terms of its normal modes y_1, y_2, y_3 and their associated frequencies $\omega_1 = 0$, $\omega_2 = \sqrt{k/m}$, $\omega_3 = \sqrt{3k/m}$, respectively. The analytic solution is analogous to the one given in (2.3) with $y_3(t)|_{space}$ having a form similar to that of $y_2(t)|_{space}$. The first mode y_1 corresponds to the displacement mode and y_2, y_3 correspond to oscillation modes with y_2 reflecting oscillations of the outer two masses while the middle one remains fixed, thus behaving like a two-mass system, and y_3 reflecting oscillations of the middle mass. The normal modes can be written more compactly in the vector form, $\boldsymbol{y} = (y_1, y_2, y_3)^T$, whose relation to x is given by the orthogonal transformation matrix V, where the columns of V correspond to the eigenvectors of the matrix $\boldsymbol{K}/m$,

$$\boldsymbol{x} = \boldsymbol{V}\boldsymbol{y}, \qquad \boldsymbol{V} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}. \tag{3.2}$$

That is to say one has $y_1 = (1/\sqrt{3})(x_1 + x_2 + x_3)$, $y_2 = (1/\sqrt{2})(x_3 - x_1)$ and $y_3 = (1/\sqrt{6})(x_1 - 2x_2 + x_3)$.

The Lagrangian function (1.1) can now be written in terms of the normal modes as follows:

$$L = \frac{1}{2}\dot{\boldsymbol{y}}^T \hat{\boldsymbol{M}} \dot{\boldsymbol{y}} - \frac{1}{2}\boldsymbol{y}^T \hat{\boldsymbol{K}} \boldsymbol{y}\ , \tag{3.3}$$

where $\hat{\boldsymbol{M}}$ is related to $\boldsymbol{M}$ via $\hat{\boldsymbol{M}} = \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V}$, yielding

$$\hat{\boldsymbol{M}} = \begin{pmatrix} m + \frac{2\epsilon^2}{3}(m_{12} + m_{13} + m_{23}) & \frac{\epsilon^2}{\sqrt{6}}(m_{23} - m_{12}) & -\frac{\epsilon^2}{3\sqrt{2}}(m_{12} - 2m_{13} + m_{23}) \\ \frac{\epsilon^2}{\sqrt{6}}(m_{23} - m_{12}) & m - \epsilon^2 m_{13} & \frac{\epsilon^2}{\sqrt{3}}(m_{12} - m_{23}) \\ -\frac{\epsilon^2}{3\sqrt{2}}(m_{12} - 2m_{13} + m_{23}) & \frac{\epsilon^2}{\sqrt{3}}(m_{12} - m_{23}) & m + \frac{\epsilon^2}{3}(-2m_{12} + m_{13} - 2m_{23}) \end{pmatrix}, \tag{3.4}$$

and $\hat{\boldsymbol{K}} = \boldsymbol{V}^T \boldsymbol{K} \boldsymbol{V}$ is diagonal with entries 0, k and $3k$, respectively. Now recall that the added masses m_{12}, m_{13} and m_{23} are functions of $x_2 - x_1$ and $x_3 - x_2$ only, which transform into the normal modes variables according to (3.2),

$$x_2 - x_1 = \frac{y_2 - \sqrt{3}y_3}{\sqrt{2}}, \qquad x_3 - x_2 = \frac{y_2 + \sqrt{3}y_3}{\sqrt{2}}. \tag{3.5}$$

To this end, one can readily write Lagrange's equations for the submerged system in terms of the normal modes. The explicit form of the resulting equations (which

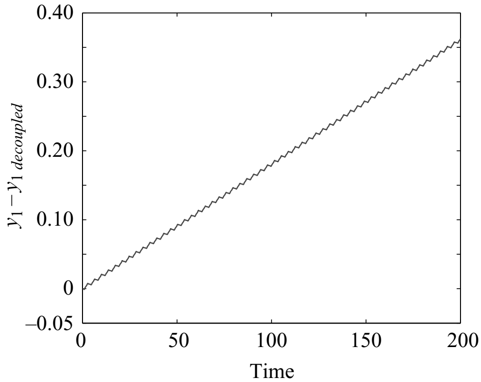

FIGURE 5. Submerged three-mass system: starting with zero displacement velocity $\dot{y}_1 = 0$, the submerged system is able to achieve a net displacement by initially exciting the oscillation mode y_3 through coupling of the normal modes. The initial conditions are $y_1 = y_2 = y_3 = 0$, $\dot{y}_1 = \dot{y}_2 = 0$ and $\dot{y}_3 = 1$. The parameter values are set to $\epsilon = 0.1$, $k = 2$, $2a^2 = 1$. The integration time is 200. The numerical solution is obtained using the following approximate expressions for the added masses: $m_{12} = -2a^2(1 + \epsilon((y_2 - \sqrt{3}y_3)/\sqrt{2}))^{-2}$, $m_{13} = -2a^2(2 + \epsilon(y_2\sqrt{2}))^{-2}$ and $m_{23} = -2a^2(1 + \epsilon((y_2 + \sqrt{3}y_3)/\sqrt{2}))^{-2}$.

are weakly nonlinearly coupled through the off-diagonal terms in $\boldsymbol{M}$) is cumbersome and therefore not shown here. We are interested in knowing whether the nonlinear coupling owing to presence of the fluid is sufficient to induce a net displacement when starting with $\dot{y}_1(0) = 0$. We emphasize here that we want to study passive displacement or locomotion; that is to say we are not interested in controlling $y_2(t)$ and $y_3(t)$ for all time but rather in gauging whether an excitation of either or both of these oscillation modes at time $t = 0$ is sufficient via fluid coupling to excite the displacement mode y_1. Remarkably, the answer is yes. Starting with $\dot{y}_1(0) = 0$ and giving an initial excitation to the oscillation mode y_3, the system is able to achieve a net passive displacement because of the coupling of these modes by virtue of the hydrodynamic effects (see figures 5 and 6). Meanwhile an initial excitation of the oscillation mode y_2 alone has no effect on the displacement mode. This should not be surprising given that when y_2 alone is excited, the three-mass system behaves as the two-mass system discussed in § 2. To verify these claims, it is instructive to examine the conserved momentum $p_1 = \partial L/\partial \dot{y}_1$ associated with the displacement mode y_1. Indeed, the relations in (3.5) imply that when viewed as functions of the normal modes, the added masses in (3.4) are independent of y_1. Consequently, the Lagrangian function in (3.3) is also independent of y_1. The variable y_1 is an ignorable coordinate, and its associated momentum $p_1 = \partial L/\partial \dot{y}_1$ is an integral of motion,

$$p_1 = \left[m + \frac{2\epsilon^2}{3}(m_{12} + m_{13} + m_{23})\right]\dot{y}_1 + \frac{\epsilon^2}{\sqrt{6}}(m_{23} - m_{12})\dot{y}_2 - \frac{\epsilon^2}{3\sqrt{2}}(m_{12} - 2m_{13} + m_{23})\dot{y}_3. \tag{3.6}$$

Equation (3.6) shows that even when $\dot{y}_1(0) = 0$, the displacement mode y_1 may be excited via its coupling with the oscillation modes y_2 and y_3 such that the value of p_1 is conserved. If y_2 is excited at time $t = 0$ while $y_3(0) = \dot{y}_3(0) = 0$, then only the second term in (3.6) is relevant, but this term is zero from symmetry (y_2 leaves the middle mass fixed and thus $m_{23} = m_{12}$); hence the result that y_2 alone has no effect on the net displacement. This result is supported by direct numerical integration of

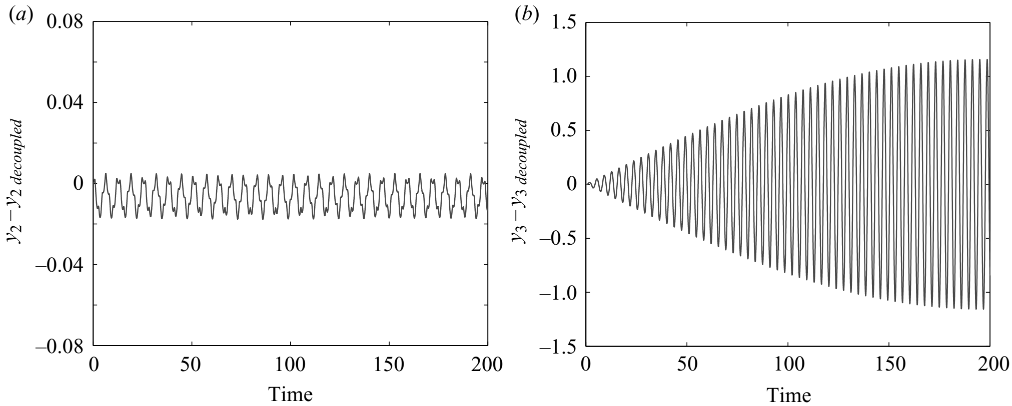

FIGURE 6. Submerged three-mass system: response of y_2 and y_3 relative to the decoupled oscillations with $\omega_2 = \sqrt{k/m}$ and $\omega_3 = \sqrt{3k/m}$ for the same parameter values and initial conditions shown in figure 5.

the equations of motion (not shown). Now, if the oscillation mode y_3 is excited at time $t = 0$, the last term in (3.6) is non-zero, which breaks the symmetry and excites the displacement mode in a way that causes the system to undergo a net passive displacement as illustrated in figure 5. Note that the underlying mechanism for the net displacement here is somewhat reminiscent of the one presented in the work of Najafi & Golestanian (2004) for three spheres connected via rigid links of time-dependent length that swim in Stokes flow by periodic contractions of these links. The main difference, besides the fluid flow regime (Stokes flow versus the potential flow employed here), is that Najafi & Golestanian (2004) actively controlled the length of the links for all time, which requires continuous energy input, whereas in this paper, we address passive locomotion in terms of initial conditions that requires an energy input only initially at time $t = 0$ to set the oscillation mode in motion; then the system evolves passively at zero energy cost.

4. Conclusions

A class of submerged mass–spring systems is proposed and analysed as a prototype for (i) studying the effect of the coupling between the system's elastic properties and the surrounding flow on the dynamic response of the system and (ii) for exploring whether passive locomotion can be achieved through such coupling. We show that with the potential flow assumption and starting with zero displacement velocity, a two-mass system cannot undergo passive locomotion. The hydrodynamic coupling, however, plays the important role of exciting the oscillation mode nonlinearly, leading to the kind of interesting forced dynamics more commonly associated with parametrically excited systems. Meanwhile, a three-mass system can move passively even when starting with zero displacement velocity. The fact that the two-mass system cannot achieve a net displacement is rooted in the time-reversibility property of the potential flow model. This reversibility property allows one to cast active swimming in terms of gauge-theoretic methods of geometric mechanics. The implications of time reversibility on designing swimming strokes at low Reynolds numbers (Stokes flow regime) is discussed in the work of Purcell (1977), and gauge-theoretic methods for swimming in this regime were pioneered by Shapere & Wilczek (1987) and have been applied since quite successfully in the potential flow regime (see, e.g., Kanso *et al.* 2005 and the references therein). These swimming models are concerned with active locomotion obtained when properly controlling for all time the internal degrees of freedom of the swimming model. In contrast, this paper is concerned with the passive dynamics of the submerged mass–spring systems and examines net displacement in terms of initial conditions only. For such freely evolving systems of submerged bodies, the gauge-theoretic methods may not be directly applicable. Indeed, in the work of Nair & Kanso (2007), we have shown that for two cylinders submerged in potential flow (not connected via a spring), when one body is forced to undergo period oscillations, the free cylinder moves away from the oscillating cylinders in a time-irreversible manner, which is a characteristic behaviour of 'dynamical systems with drift'.

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